Math 102

Krishanu Sankar

September 20, 2018

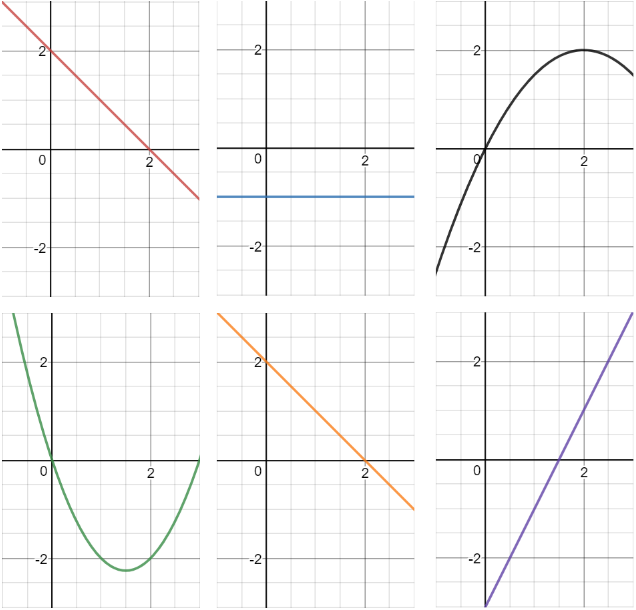
Announcements

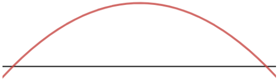
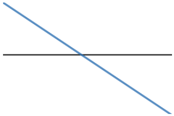
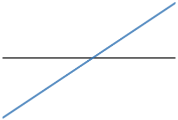
- Next week - I will be away. Colin Macdonald (section 104) will substitute on Tuesday and Thursday.
- Reminders - MLC, Piazza

Goals Today

- The derivative, graphically
  - Critical points, local minima/maxima
  - $f''(x)$ and inflection points
  - Graph sketching
- Some rules of differentiation
  - The Power Rule
  - The Product Rule and Quotient Rule

Last Time: Matching



$f(x)$ looks like	$f'(x)$ looks like
	
	
	
	

Critical Points

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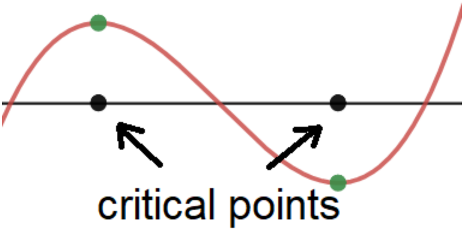

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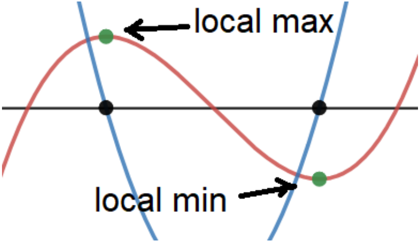
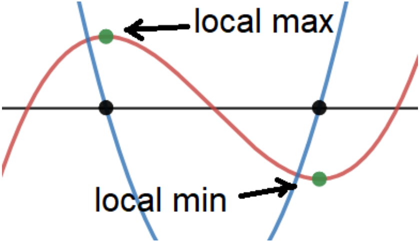


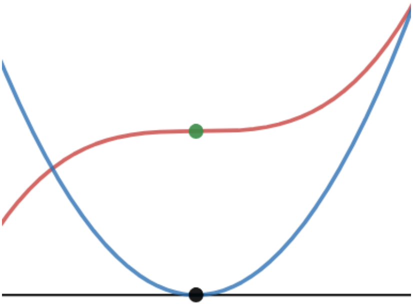
- **Local maximum:** occurs when the graph of $f'(x)$ goes from positive to negative.
- **Local minimum:** occurs when the graph of $f'(x)$ goes from negative to positive.

Critical Points



- **Local maximum:** occurs when the graph of $f'(x)$ goes from positive to negative.
- **Local minimum:** occurs when the graph of $f'(x)$ goes from negative to positive.
- Question: Can you have a critical point which is neither a local minimum nor local maximum?

Answer: Yes! This will happen when the graph of $y = f'(x)$ reaches zero but does not cross the axis.



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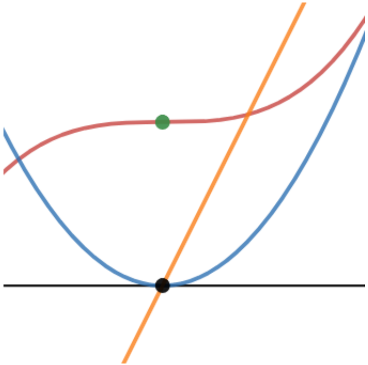

$y = f''(x)$ in orange. In this case, $f''(x)$ is a linear function, $f'(x)$ is a degree 2 polynomial, and $f(x)$ is a degree 3 polynomial.

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- It measure the **concavity** (i.e., ‘curvedness) of the graph of $y = f(x)$.

$$f''(x) > 0 \implies \text{concave up (like a cup)}$$
$$f''(x) < 0 \implies \text{concave down (like a frown)}$$
$$f''(x) = 0 \implies \textbf{inflection point}$$

- https://www.desmos.com/calculator/unskahz0pf

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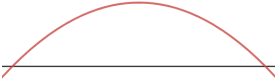	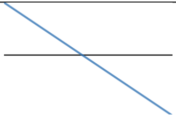	
	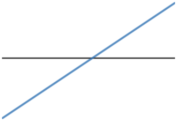	
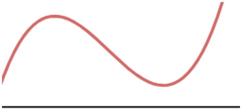	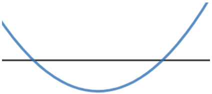	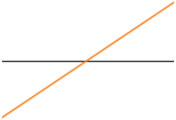

Position, velocity, acceleration

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- Let $x(t)$ be the **position** of an object at time t.
- $x'(t) = v(t)$ is the **velocity**.
- $x''(t) = v'(t) = a(t)$ is the **acceleration**.

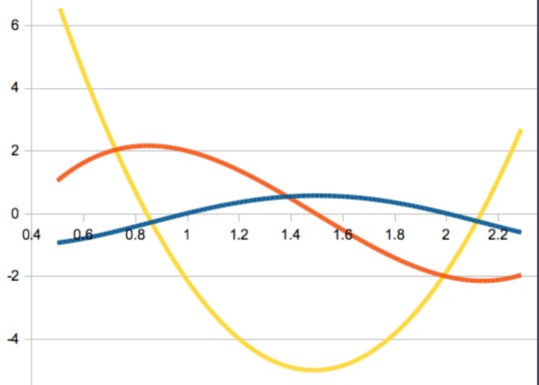
Which is x, v, a?

(A) x, v, a

(B) x, v, a

(C) x, v, a

(D) x, v, a



Check max/mins --> zeros, check inc/dec --> +/-.

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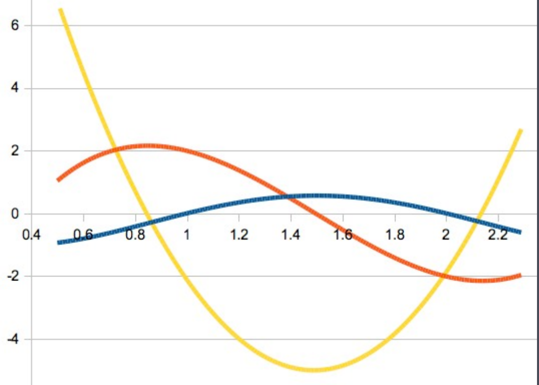
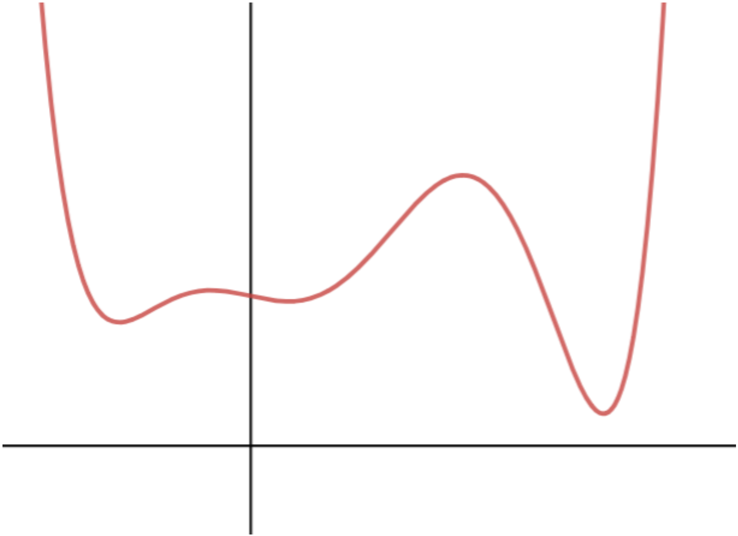


Check max/mins --> zeros, check inc/dec --> +/-.

Example on the board: given the graph of $f(x)$, sketch a graph of $f'(x)$.

Now you try!



Visually sketching the derivative

Strategy:

- Identify the **critical points**
- Determine whether the derivative is **positive or negative** in between
- (Optional, for more accuracy) Identify the **inflection points**
- Sketch! Remember, **steeper = bigger derivative**
  'bigger' means in absolute value

The Power Rule

$f(x)$	$f'(x)$
x	1
x^2	$2x$
x^3	$3x^2$
$\vdots$	$\vdots$
x^n	nx^{n-1}

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$f(x)$	$f'(x)$
$\sqrt{x} = x^{1/2}$	$\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$
$\sqrt[3]{x} = x^{1/3}$	$\frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$
$1/x = x^{-1}$	$-x^{-2} = -1/x^2$

This works for negative and fractional powers too. For example, if $f(x) = \sqrt{x} = x^{1/2}$, then $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$. (Proof on board)

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Exercise: **Using the definition as a limit,** prove that if $f(x) = 1/x$, then $f'(x) = -1/x^2$.

Bonus Slide - $f(x) = \sqrt{x}$

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
\end{aligned}
$$

Bonus Slide - $f(x) = \frac{1}{x}$

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h \to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} \\
&= \lim_{h \to 0} \frac{\frac{-h}{x(x+h)}}{h} = \lim_{h \to 0} \frac{-h}{hx(x+h)} \\
&= \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}
\end{aligned}
$$

Exercise: For each function below, write the equation of the tangent line at the given point.

- Tangent to $y = x^2$ through the point $(-3, 9)$
- Tangent to $y = \frac{6}{x}$ through the point $(2, 3)$

Hint: The line through (x_0, y_0) with slope m is given by the equation

$$y - y_0 = m(x - x_0)$$

Bonus Slide - $y = x^2$

$$f(x) = x^2 \implies f(-3) = 9$$

$$f'(x) = 2x \implies f'(-3) = -6$$

So the tangent line at $(-3, 9)$ has equation $y - 9 = -6(x + 3)$, which can be rewritten as

$$y = -6x - 9$$

Bonus Slide - $y = \frac{6}{x}$

$$f(x) = \frac{6}{x} \implies f(2) = 3$$

$$f'(x) = -\frac{6}{x^2} \implies f'(2) = -\frac{3}{2}$$

So the tangent line at $(2, 3)$ has equation $y - 3 = -\frac{3}{2}(x - 2)$, which can be rewritten as

$$y = -\frac{3}{2}x + 6$$

Product and Quotient Rules

- The Product Rule

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

  has a proof by picture - https://www.desmos.com/calculator/w3uck8beru
- The Quotient Rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Recap

- The derivative, graphically
  - Critical points, local minima/maxima
  - $f''(x)$ and inflection points
  - Graph sketching
- Some rules of differentiation
  - The Power Rule
  - The Product Rule and Quotient Rule

Bonus - Proof of Product Rule

Note: The following proof is purely for your viewing pleasure, and is not something you will be evaluated on!

Let $P(x)$ denote the function $P(x) = f(x)g(x)$. Then

$$\begin{aligned} P'(x) &= \lim_{h \to 0} \frac{P(x+h) - P(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \end{aligned}$$

Bonus - Proof of Product Rule

Let's create some notation. Write
$\Delta f = f(x+h) - f(x)$ and $\Delta g = g(x+h) - g(x)$.
By checking the algebra, you will see that
$f(x+h)g(x+h) - f(x)g(x)$ is equal to

$$(\Delta f)(\Delta g) + (\Delta f)g(x) + f(x)(\Delta g)$$

Thus,

$$P'(x) = \lim_{h\to 0} \frac{(\Delta f)(\Delta g) + (\Delta f)g(x) + f(x)(\Delta g)}{h}$$

Bonus - Proof of Product Rule

$$P'(x) = \lim_{h\to 0} \frac{(\Delta f)(\Delta g) + (\Delta f)g(x) + f(x)(\Delta g)}{h}$$

But

$$\lim_{h\to 0} \frac{(\Delta f)g(x)}{h} = g(x) \lim_{h\to 0} \frac{\Delta f}{h} = g(x)f'(x)$$

$$\lim_{h\to 0} \frac{f(x)(\Delta g)}{h} = f(x) \lim_{h\to 0} \frac{\Delta g}{h} = f(x)g'(x)$$

$$\lim_{h\to 0} \frac{(\Delta f)(\Delta g)}{h} = (\lim_{h\to 0} \Delta f)\cdot(\lim_{h\to 0} \frac{\Delta g}{h}) = 0\cdot g'(x) = 0$$